HW 2 P 89

(1a) $\exp(2 \pm 3\pi i) = e^2 e^{\pm\pi i} = -e^2$

(1b) $\exp\left(\frac{2+\pi i}{4}\right) = e^{1/2} e^{\pi/4 i} = \sqrt{\frac{e}{2}}(1+i)$

(1c) $\exp(z+\pi i) = -\exp(z)$

(5) $|\exp(2z+i)| = |\exp(2z)|$

$= e^{2x}$

$|\exp(i(x^2-y^2+2ixy))| = e^{-2xy}$

$|\exp(2z+i) + \exp(iz^2)| \le |\exp(2z+i)| + |\exp(iz^2)|$

$\le e^{2x} + e^{-2xy}$

(6) $\exp(z^2) = \exp(x^2-y^2+2ixy)$

$|\exp(z^2)| = \exp(x^2-y^2)$

$\exp(|z|^2) = \exp(x^2+y^2)$

$\Rightarrow |\exp(z^2)| \le \exp(|z|^2)$.

(7) if $|\exp(-2z)| < 1$

$\exp(-2x) < 1$

then $x > 0$, the reverse direction is the same.

8a $e^{x+yi} = -2$

$= -2e^{\pi i + 2n\pi i}$

$\Rightarrow z = \log 2 + i(\pi + 2\pi n) \quad n \in \mathbb{Z}$.

8b $e^{x+yi} = 1 + i = \sqrt{2}\, e^{\pi/4 i + 2\pi n i}$

$z = \frac{1}{2}\log 2 + \left(2n + \frac{1}{4}\right)\pi i, \quad n \in \mathbb{Z}$

8c $\exp(2z-1) = 1$

$e^{2x-1+2yi} = e^{0+2\pi n i}$

$z = \frac{1}{2} + n\pi i, \quad n \in \mathbb{Z}$.

P95

1a $\mathrm{Log}(-ei) = \mathrm{Log}\left(e \cdot e^{-\pi/2 i}\right) = 1 - \pi/2\, i$

1b $\mathrm{Log}(1-i) = \mathrm{Log}\left(\sqrt{2}\, e^{-\pi/4 i}\right)$

$= \frac{1}{2}\log 2 - \pi/4\, i$

2a $\log e = \log\left(e\, e^{2n\pi i}\right) \quad n \in \mathbb{Z}$

$= 1 + 2\pi n i$

2b $\log i = \log\left(e^{(\pi/2 + 2\pi n)i}\right)$

$= \left(2n + 1/2\right)\pi i \quad n \in \mathbb{Z}$.

2c $\log(-1+\sqrt{3}i) = \log\left(2\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right)$

$= \log 2 + \log e^{(2n\pi + 2/3\pi)i} \quad n \in \mathbb{Z}$

$= \log 2 + 2(n+1/3)\pi i$

3 $\operatorname{Log}(i^3) = \operatorname{Log}(-i) = \operatorname{Log}(e^{-\pi/2 i}) = -\pi/2\, i$

$3 \operatorname{Log} i = 3 \operatorname{Log}(e^{\pi/2 i}) = \frac{3\pi}{2} i$

$\therefore \operatorname{Log}(i^3) \neq 3 \operatorname{Log} i$

4 $\log(i^2) = \log(-1) = 3\pi/2\, i$

$\log(i) = \log(e^{\frac{5\pi}{2}i}) = \frac{5\pi}{2} i$

$\therefore \log(i^2) \neq 2 \log i$

5 a $z^2 = i = e^{\pi/2 i}$

$z = e^{(\pi/4 + n\pi)i} \quad n = 0, 1$

$= e^{\pi/4 i}, e^{\frac{5\pi}{4}i}$

$\log(e^{i\pi/4}) = \log(e^{i\pi/4 + 2n\pi i}) = \left(2n + \frac{1}{4}\right)\pi i, \ n \in \mathbb{Z}$

Similarly, $\log(e^{5\pi/4 i}) = \left((2n+1) + \frac{1}{4}\right)\pi i$

Since $\{2n\}$ and $\{2n+1\}$ form the set of integers.

Thus $\log(i^{1/2}) = (n + 1/4)\pi i, \ n \in \mathbb{Z}$.

5b $\log i = \log\left(e^{\pi/2\, i + 2n\pi i}\right)$

$= 2\left(n + \frac{1}{4}\right)\pi i \quad . \; n \in \mathbb{Z}.$

$= 2\log(i^{1/2})$ by 5a.

8 let $z = re^{i\theta}$ with $\theta \in (-\pi, \pi]$

$\log z = \log r + (2\pi n + \theta) i, \; n \in \mathbb{Z}$

$= i\pi/2$

$\Rightarrow r = 1$ and $n = 0, \; \theta = \pi/2$

Thus, $z = i$.

9 Since $e^z = e^{x+iy}$

$\log e^z = \log e^x + iy$ (since $\alpha < y < \alpha + 2\pi$)

$= x + iy = z.$